EXERCISE SHEET FOR THE PHD COURSE "INTRODUCTION TO BOUNDED COHOMOLOGY AND SIMPLICIAL VOLUME"

Exercise 1. Let X be a path-connected topological space. The aim of this exercise is to show that there exists a well-defined isomorphism

$$H^1(X;\mathbb{R}) \to \mathrm{Hom}(\pi_1(X), \mathbb{R}) \ .$$

Here, we interpret a cocycle $\varphi \in C^1(X;\mathbb{R})$ as a function from paths in X to $\mathbb{R}$. To this end it is convenient to prove the followings:

(i) $\varphi(\gamma * \eta) = \varphi(\gamma) + \varphi(\eta)$, where γ and η are to consecutive paths in X and $*$ denotes the concatenation of paths;
(ii) $\varphi(c) = 0$ for all constant paths c in X;
(iii) $\varphi(\gamma) = \varphi(\eta)$ if γ and η are homotopic relative to their endpoints;
(iv) φ is a coboundary if and only if $\varphi(\gamma)$ only depends on the endpoints of γ;
(v) $\mathrm{Hom}(\pi_1(X), \mathbb{R}) \cong \mathrm{Hom}(H_1(X;\mathbb{Z}), \mathbb{R})$.

Exercise 2. The aim of this exercise is to compute the 0-th and the 1-st bounded cohomology groups of $\mathbb{S}^1$ with $\mathbb{R}$-coefficients.

(i) Compute $H_b^0(\mathbb{S}^1;\mathbb{R})$;
(ii) Prove that the comparison map $c^1_{\mathbb{S}^1} \colon H_b^1(\mathbb{S}^1;\mathbb{R}) \to H^1(\mathbb{S}^1;\mathbb{R})$ is the zero map (*Hint:* The Kronecker product might be useful);
(ii) Let $\varphi \in C_b^1(\mathbb{S}^1;\mathbb{R})$ be such that $\varphi = \partial f$ with $f \in C^0(\mathbb{S}^1;\mathbb{R})$. Prove that, there exists a $\overline{f} \in C_b^0(\mathbb{S}^1;\mathbb{R})$ such that $\varphi = \partial \overline{f}$.

Exercise 3. Compute the 0-th and the 1-st bounded cohomology groups of any path-connected topological space. (*Hint:* Hurewicz Theorem and Exercise 1 might be useful.)

Exercise 4. Let $(V, \|\cdot\|)$ be a normed vector space. Let W be a linear subspace of V. Then, we define a function

$$\rho \colon V/W \to \mathbb{R}$$

by

$$\rho([v]) = \rho(v + W) \coloneqq \inf\{\|v + w\| \mid w \in W\} \ ,$$

where $[v] \in V/W$ and $v \in V$.

(i) Show that ρ is a norm if and only if W is closed;
(ii) Assume that V is Banach normed vector space. Show that V/W is Banach normed vector space if and only if W is closed.

Exercise 5. Show that the inhomogeneous chain complex (i.e. the one associated to the bar resolution) is chain isomorphic to the homogeneous one.

Definition 0.1. Let $F_2 = \langle a, b \rangle$ be the free non-Abelian group of rank 2 and let

$$\mathrm{Rol} \coloneqq \{\alpha \colon \mathbb{Z} \to \mathbb{R} \,|\, \|\alpha\|_\infty < +\infty, \alpha(n) = -\alpha(-n) \text{ for all } n \in \mathbb{Z}\} \ .$$

For every $\alpha \in \mathrm{Rol}$ we consider the map

$$q_\alpha \colon F_2 \to \mathbb{R}$$

defined by

$$f_\alpha(a^{n_1} b^{m_1} \cdots a^{n_k} b^{m_k}) = \sum_{i=1}^{k} f(n_i) + f(m_i) \ ,$$

where we are identifying each element of F_2 with the unique word in reduced form representing it (we allow n_1 and m_k to be 0).

The map q_α is called *Rolli's quasi-morphism.*

Definition 0.2. Let $F_2 = \langle a, b \rangle$ be the free non-Abelian group of rank 2 and let $\omega \in F_2$ be a reduced word. We define

$$q_\omega \colon F_2 \to \mathbb{R}$$

by

$$q_\omega(g) = \# \text{ occurencies of } \omega \text{ in } g - \# \text{ occurencies of } \omega^{-1} \text{ in } g \ ,$$

where g is written in reduced form (e.g. if $\omega = abab$, then $q_\omega(ababab) = 2 - 0 = 2$).

The map q_ω is called *Brooks quasi-morphism.*

Exercise 6. Show that

(i) For every $\alpha \in \mathrm{Rol}$, q_α is a quasi-morphism and compute its defect.
(ii) For every reduced word $\omega \in F_2$, q_ω is a quasi-morphism and compute its defect.

Exercise 7. Prove that $F_2 = \langle a, b \rangle$ is not amenable (using the definition of amenable groups via left-invariant means).

Exercise 8. Let M be a n-dimensional manifold and let V an m-dimensional submanifold of M.

- Prove that if $m = n - 2$ the inclusion:

$$i \colon M \setminus V \to M$$

induces a surjection at the level of fundamental groups:

$$\pi_1(M \setminus V) \to \pi_1(M).$$

- Let Σ_g be a surface of genus $g \geq 2$. Use the previous result to construct an epimorphism $\pi_1(\Sigma_g) \to F_g$, where F_g denotes the non-abelian free group of rank g.
- Conclude that $H_b^2(\pi_1(\Sigma_g); \mathbb{R})$ is infinite dimensional, where $\mathbb{R}$ is endowed with the trivial action.

Exercise 9. Prove that

(i) A chain cocomplex $(C^k, \delta^k)_{k\geq 0}$ of free modules over $\mathbb{R}$ is acyclic (i.e. $\mathrm{Im}(\delta^k) = \ker(\delta^{k+1})$) if and only if there exists a family of maps $h_k \colon C^k \to C^{k-1}$ such that

$$\delta^{k-1} \circ h_k + h_{k+1} \circ \delta^k = \mathrm{Id}_{C^k}.$$

Let Γ be a discrete group and let V be an Γ-module over $\mathbb{R}$. Show that

(ii) The complex $(C^\bullet(\Gamma, V))$ is acyclic by constructing a family of maps as above (we have already seen several proofs of this fact!).

Definition 0.3. A family $\{G_i\}_{i\in I}$ is a *direct system* if for every $i, j \in I$ there exists $k \in I$ such that $G_i, G_j < G_k$.

The *direct union* (or *direct limit*) of a *direct system* $\{G_i\}_{i\in I}$ is the group whose underlying set is

$$G := \bigsqcup_{i\in I} G_i / \sim$$

where for each $g \in G_i$ and $h \in G_j$ we have $g \sim h$ with $G_i, G_j < G_k$ then $g = h$ in G_k. The composition of $g \in G_i$ and $h \in G_j$ is $gh \in G_k$.

Exercise 10. Prove that the direct union of amenable groups is amenable.

(*Hint:* Consider the set of measures on the direct union that are invariant with respect to the action of the groups in the direct system. Then use compactness of the set of measures with respect to the weak* topology to show that the intersection of such sets is non-empty.)

Definition 0.4. A group Γ is *virtually solvable* if it contains a solvable subgroup of finite-index .

Definition 0.5. A group Γ is *residually finite* if for each element $g \neq e \in \Gamma$ there exists a finite-index normal subgroup $N(g) \trianglelefteq \Gamma$ which does not contain g.

Exercise 11. Prove that virtually solvable groups are amenable.

Exercise 12. Show that

(i) A group is a subgroup of an infinite direct product of non-trivial finite groups if and only if it is residually finite (*Hint:* it could be useful to express the notion of residually finite groups in terms of certain homomorphisms onto finite groups).

(ii) (*Bonus*) Free groups are residually finite.

Conclude (using item *(ii)* is allowed) that

(iii) In general, the infinite direct product of amenable groups is not amenable.

Definition 0.6. A discrete countable group Γ satisfies the *Følner condition* if for every finite subset $A \subset \Gamma$ and every $\varepsilon > 0$ there exists a finite non-empty subset $F \subset \Gamma$ such that for each $a \in A$ we have

$$\frac{|aF \bigtriangleup F|}{|F|} \leq \varepsilon \ ,$$

where $\bigtriangleup$ denotes the symmetric difference of sets.

Given a discrete and countable group Γ, a *Følner sequence* is a sequence $\{F_n\}_{n\in\mathbb{N}}$ of non-empty and finite subsets of Γ such that

$$\frac{|gF_n \bigtriangleup F_n|}{|F_n|} \to 0 \ ,$$

for every $g \in \Gamma$.

Exercise 13. Prove that

(i) A discrete countable group Γ satisfies the Følner condition if and only if it has a Følner sequence;
(ii) $\mathbb{Z}$ has a Følner sequence;
(iii) Every finitely generated Abelian group has a Følner sequence;
(iv) If every finitely generated subgroup of a discrete countable group Γ satisfies the Følner condition, then Γ does so;
(v) All Abelian groups satisfy the Følner condition.
(vi) *Bonus:* Provide some intuitive explanations that the free groups of rank at least 2 does not have any Følner sequence.

Remark 0.7. One can prove that a group Γ is amenable if and only if it satisfies the Følner condition (or, equivalently, it has a Følner sequence). So the previous exercise gives a proof of the mentioned fact that Abelian groups are amenable. Moreover, we provided a new proof of the fact that free groups of rank at least 2 are not amenable.

Exercise 14. Show that a group Γ is amenable if and only if the trivial normed Γ-module $\mathbb{R}$ over $\mathbb{R}$ is relatively injective. (*Hints:* ($\Rightarrow$) It may be easier to prove that every dual normed Γ-modules over $\mathbb{R}$ is relatively injective. ($\Leftarrow$) It may be useful to use the characterization of amenability in terms of the existence of an invariant continuous non-trivial functional on $\ell^\infty(\Gamma)$.)

Exercise 15. Find a paradoxical decomposition of F_2.

Exercise 16. Prove that the following conditions are equivalent.

- Γ is finite;
- $H_b^n(\Gamma, V) = 0$ for every $\mathbb{R}[\Gamma]$-module V and every $n \geq 1$.
- $H_b^1(\Gamma, V) = 0$ for every $\mathbb{R}[\Gamma]$-module V.

Exercise 17. Let V be a Banach Γ-module over $\mathbb{R}$ and let $(V, V^\bullet, \delta^\bullet)$ be any strong resolution of V. Then, the identity over V can be extended to a chain map $\alpha^\bullet$ between $V^\bullet$ and the standard resolution of V in such a way that $\|\alpha^n\| \leq 1$ for every $n \geq 0$.

As we mentioned in the lecture the extension can be constructed as follows: for every $v \in V^n$ and $g_0, \cdots, g_n \in \Gamma$ we set

$$\alpha^n(v)(g_0, \cdots, g_n) = \alpha^{n-1}(g_0(k^n(g_0^{-1}(v))))(g_1, \cdots, g_n) \ .$$

(i) Prove that it is a chain Γ-map (recall that we have already proved during the lecture that it is norm non-increasing).

Exercise 18. Let V be a relatively injective normed Γ-module over $\mathbb{R}$, let $j\colon A \to B$ be a Γ-map (i.e. bounded and Γ-equivariant) and suppose that there exists an $\mathbb{R}$-linear map $\sigma\colon B \to A$ such that $\|\sigma\| \le 1$ and $j \circ \sigma \circ j = j$. Let also $\alpha\colon A \to V$ be a Γ-map and suppose that $\ker(j) \subset \ker(\alpha)$.

(i) Prove that there exists a Γ-map $\beta\colon B \to V$ such that $\beta \circ j = \alpha$ and $\|\beta\| \le \|\alpha\|$.

Exercise 19. Let $\mathrm{id}\colon V \to V$ be a Γ-map between Banach Γ-modules. Let $(V, V^\bullet, \delta_V^\bullet)$ be a strong resolution of V and suppose that $(W, W^\bullet, \delta_W^\bullet)$ is relatively injective resolution of V. Then, id extends to a chain Γ-map $\alpha^\bullet$ (as we proved in the lecture).

(i) Prove that any two such extensions of id to chain maps are Γ-homotopic.

Exercise 20. Compute the integral simplicial volume of n-spheres and of oriented closed connected surfaces.

Exercise 21. Let M and N be two oriented closed connected manifold of dimension m, n respectively. Prove that

$$\|M\| \cdot \|N\| \le \|M \times N\| \le \binom{n+m}{m} \|M\| \cdot \|N\| \ .$$

Exercise 22. Prove that

(i) The function $\|\cdot\|_{\mathbb{Z}}\colon H_n(M, \partial M; \mathbb{Z}) \to \mathbb{R}$ is not a semi-norm.

Discuss

(ii) The relation between the integral simplicial volume of the base space N and the covering space M for a degree d covering map $f\colon M \to N$.

Assume to know that $\|\Sigma_g\| \ge 2|\chi(\Sigma_g)|$.

(iii) Prove that $\|\Sigma_g\| = 2|\chi(\Sigma_g)|$.

Exercise 23. Let $p\colon E \to B$ be a locally trivial fiber bundle with fiber F, where F, E, B are orientable compact connected manifolds. Assume $\dim(F) \ge 1$ and that $i_*(\pi_1(F)) \le \pi_1(E)$ of the fundamental group of F under the inclusion $i\colon F \hookrightarrow E$ is amenable. Prove that

(i) $\|E\| = 0$.

Then,

(ii) Construct explicitly an example of such manifold E.

Theorem 0.8 (Gromov's Proportionality Principle)**.** *Let M and N be n-manifolds which share the same Riemannian universal covering. Then,*

$$\frac{\|M\|}{\mathrm{vol}(M)} = \frac{\|N\|}{\mathrm{vol}(N)} \ .$$

Exercise 24. Prove Gromov's Proportionality Principle 0.8 in the case in which the isometry group of $\widetilde{M} \cong \widetilde{N}$ is discrete.